Springer

Journal of Mechanical Science and Technology

Journal of Mechanical Science and Technology 23 (2009) 1058~1064

www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-009-0341-4

Dynamic behaviors of nonlinear fractional-order differential oscillator †

Wei ZHANG[1,*], Shao-kai LIAO[2] and Nobuyuki SHIMIZU [3]

[1]*Department of Electronic Engineering, JiNan University, Guangzhou, China*
[2]*Institute of Mechanical Science and Technology, Shenzhen, China*
[3]*Department of Mechanical and Power Engineering, Iwaki Meisei University, Japan*

(Manuscript Received December 24, 2008; Revised March 16, 2009; Accepted March 16, 2009)

Abstract

The nonlinear dynamic behaviors of oscillators described by fractional-order differential are presented in this paper. The background of the research is based upon two engineering practices. First is that the visco-elastic behaviors of some advanced polymeric materials can be accurately modeled by the fractional calculus constitutive law. Second is that the influence of nonlinear visco-elasticity described by the fractional operator cannot be neglected in some cases such as the vibration with large displacement or large strain and thermo-visco-elastic coupled problems. The numerical scheme for solving the nonlinear equation of motion is developed. The results show that because of the introduction of nonlinear damping modeled by the fractional-order operator, the bifurcation and chaos of the oscillator appear in forced vibration. Furthermore, the fraction value of the fractional operator evidently affects the dynamic behavior of the nonlinear fractional differential oscillator.

Keywords: Nonlinearity; Fractional calculus; Bifurcation; Chaos; Viscoelasticity

1. Introduction

The nonlinear behaviors of the mechanical systems described by nonlinear fractional calculus have begun to appear in literature in recent years due to the development of fractional calculus and fractals. Addolfsson, Enelund, and Larsson (2005) studied the models and numerical procedures for nonlinear fractional order viscoelastics fractional differentiation [1]. Meshaka, Stephane, and Christian (2005) demonstrated how describing the viscoelasticity by using thermodynamic functions is linked with fractional type operators [2], while Nasuno and Shimizu (2005) presented the experimental evidence of nonlinear static and dynamic models of a fractional derivative viscoelastic body [3]. These studies remind us that the nonlinear influence on both the constitutive relationship of materials and mechanical behavior of structures cannot be ignored in some cases of engineering applications [6, 8]. It is therefore necessary that we expose, understand, and simulate the nonlinear characteristics of the dynamic systems described by a fractional-differential operator.

Based on the studies on viscoelasticity of polymer materials [3, 4], Zhang and Shimizu proposed a nonlinear viscoelastic constitutive model with a fractional-differential operator

$$\sigma(t) = q_0\varepsilon(t) + D^q\left(q_2\left[\varepsilon(t)\right]^2 + q_1\varepsilon(t)\right) \quad 0 < q < 1 \tag{1}$$

where $D^q(\cdot)$ is the Riemann-Liouville fractional derivative operator; $\sigma(t)$ is the stress; $\varepsilon(t)$ is the strain; and q_0, q_1, *and* q_2 are the model parameters. Thus, we can obtain the four-parameter nonlinear

† This paper was presented at the 4th Asian Conference on Multibody Dynamics(ACMD2008), Jeju, Korea, August 20-23, 2008.

*Corresponding author. Tel.: +86 20 85220415, Fax.: +86 20 85220415
E-mail address: tzhangw@jnu.edu.cn

fractional Kelvin-type viscoelastic model. By adopting the nonlinear constitutive law (1) to the oscillator system, we can obtain a type of a simple nonlinear dynamic system of a structure described by a fractional-differential operator

$$D^2x(t)+D^q(c_2[x(t)]^2+c_1x(t))+kx(t)=f(t) \tag{2}$$

where *x(t) denotes* the non-dimensional displacement of the oscillator, *f(t)* denotes the outside excitation, *k* describes the stiffness of the oscillator, and c_1 and c_0 refer to the linear and nonlinear damping, respectively, acted on the oscillator. A special case of a Van der pol-like fractional oscillator can be assumed as

$$D^2x(t)+\varepsilon(1-[x(t)^2])D^qx(t)+x(t)+x(t)=f(t) \tag{3}$$

where ε is a non-dimensional positive real number.

By referring to (1), (2), and (3), this paper presents the results of the nonlinear dynamic behaviors of the oscillators described by fractional-differential equations. The numerical schemes for solving the nonlinear Eqs. (2) and (3) are developed, which are based on the Zhang-Shimizu numerical algorithm of fractional derivative [5, 7]. The dynamic behaviors of the proposed nonlinear oscillator system are compared with the classical correspondent nonlinear oscillators. The results show that because of the nonlinear damping modeled by the fractional-order operator, the bifurcation and even the chaos of the oscillators appear in the forced vibration. Furthermore, the fraction value of the fractional operator evidently affects the dynamic behavior of the nonlinear fractional oscillator.

2. Numerical algorithm of fractional derivatives

The Riemann-Liouville fractional derivative is defined as

$$\begin{cases} D^q f(t)=\dfrac{1}{\Gamma(1-q)}\dfrac{d}{dx}\displaystyle\int_0^t \dfrac{f(\tau)}{(t-\tau)^q}d\tau \\ 0<q<1 \end{cases} \tag{4}$$

where $\Gamma(\bullet)$ denotes the Gamma function.

By replacing *f(t)* with $[x(t)]^2$ in Eq. (4) and assuming at least the first continuous derivative of *x(t),* we can obtain

$$\begin{cases} D^q[x(t)]^2=\dfrac{1}{\Gamma(1-q)}\{\dfrac{[x(0)]^2}{t^q}+\displaystyle\int_0^t \dfrac{2x(t)\dot{x}(t)}{(t-\tau)^q}d\tau\} \\ 0<q<1 \end{cases} \tag{5}$$

From Zhang-Shimizu's numerical scheme [5, 7], we can write

$$\begin{aligned} D^q([x(t_n)]^2) &= \frac{1}{\Gamma(1-q)}\left(\frac{[x(0)]^2}{t_n^q}+\int_0^{t_n}\frac{2x(\tau)\dot{x}(\tau)}{(t_n-\tau)^q}d\tau\right) \\ &= \frac{2}{\Gamma(1-q)}\left(\frac{[x(0)]^2}{2t_n^q}+\int_0^{t_{n-1}}\frac{x(\tau)\dot{x}(\tau)}{(t_n-\tau)^q}d\tau+\int_{t_{n-1}}^{t_n}\frac{x(\tau)\dot{x}(\tau)}{(t_n-\tau)^q}d\tau\right) \\ &= \frac{2}{\Gamma(1-q)}(J_0+J_{n-1}+\Delta J_n) \end{aligned} \tag{6}$$

where

$$J_0=\frac{[x(0)]^2}{2t_n^q} \quad \text{and} \quad \Delta J_n=\int_{t_{n-1}}^{t_n}\frac{x(\tau)\dot{x}(\tau)}{(t_n-\tau)^q}d\tau$$

If the trapezoidal integration method is adopted to calculate

$$J_{n-1}=\int_0^{t_{n-1}}\frac{x(\tau)\dot{x}(\tau)}{(t_n-\tau)^q}d\tau ,$$

we can have

$$\begin{aligned} J_{n-1} &= \int_0^{t_{n-1}}\frac{x(\tau)\dot{x}(\tau)}{(t_n-\tau)^q}d\tau \\ &= \frac{\Delta t}{2}\left[\frac{x_0\dot{x}_0}{t_n^q}+\frac{x_{n-1}\dot{x}_{n-1}}{\Delta t^q}+2\sum_{i=1}^{n-2}\frac{x(i\Delta t)\dot{x}(i\Delta t)}{(t_n-i\Delta t)^q}\right] \end{aligned} \tag{7}$$

For ΔJ_n, suppose that Eq. (7) is

$$\dot{x}(\tau)=\dot{x}_{n-1}+\eta(\tau)(\dot{x}_n-\dot{x}_{n-1}) \tag{8}$$

where $\eta(\tau)=\dfrac{\tau-t_{n-1}}{\Delta t}$, $\Delta t=t_n-t_{n-1}$, $t_{n-1}<\tau<t_n$.

If we further take

$$\ddot{x}_n=\frac{1}{\beta\Delta t^2}(x_n-x_{n-1})-\frac{1}{\beta\Delta t}\dot{x}_{n-1}-\left(\frac{1}{2\beta}-1\right)\ddot{x}_{n-1} \tag{9}$$

$$\dot{x}_n=\dot{x}_{n-1}+(1-\alpha)\Delta t\ddot{x}_{n-1}+\alpha\Delta t\ddot{x}_n \tag{10}$$

where α and β are the parameters of the Newmark-type numerical integration.

By substituting Eqs. (8) and (9) for Eq. (10), we can derive

$$\dot{x}(\tau)=\dot{x}_{n-1}+\left[(1-\alpha)\ddot{x}_{n-1}+\alpha\ddot{x}_n\right](\tau-t_{n-1}) \tag{11}$$

Likewise, by substituting Eqs. (9) and (11) for ΔJ_n we can obtain

$$\Delta J_n = \frac{\dot{x}_{n-1} x_{n-1} \Delta t^{1-q}}{1-q} + \frac{\dot{x}_{n-1}^2 \Delta t^{2-q}}{(1-q)(2-q)} + \frac{\left[(1-\alpha)\ddot{x}_{n-1} + \alpha \ddot{x}_n\right] x_{n-1} \Delta t^{2-q}}{(1-q)(2-q)} + \frac{3\left[(1-\alpha)\ddot{x}_{n-1} + \alpha \ddot{x}_n\right] \dot{x}_{n-1} \Delta t^{3-q}}{(1-q)(2-q)(3-q)} + \frac{3\left[(1-\alpha)\ddot{x}_{n-1} + \alpha \ddot{x}_n\right]^2 \Delta t^{4-q}}{(1-q)(2-q)(3-q)(4-q)} \quad (12)$$

By substituting J_0, J_{n-1}, and ΔJ_n for Eq. (6), we can then find

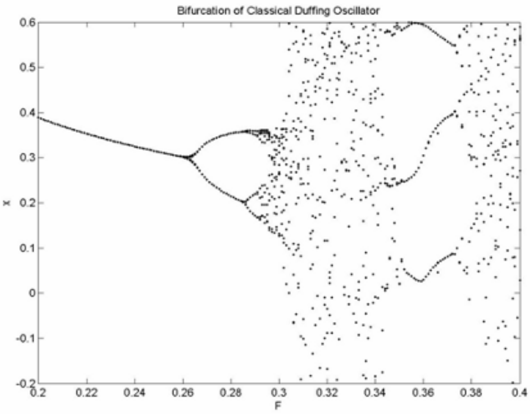
$$D^q\left(\left[x(t_n)\right]^2\right) = \frac{2}{\Gamma(1-q)} \left\{ \begin{aligned} & \frac{[x(0)]^2}{2t_n^q} + \frac{\Delta t}{2}\left[\frac{x_0 \dot{x}_0}{t_n^q} + \frac{x_{n-1}\dot{x}_{n-1}}{\Delta t^q} + 2\sum_{i=1}^{n-2} \frac{x(i\Delta t)\dot{x}(i\Delta t)}{(t_n - i\Delta t)^q}\right] \\ & + \frac{\dot{x}_{n-1} x_{n-1} \Delta t^{1-q}}{1-q} + \frac{\dot{x}_{n-1}^2 \Delta t^{2-q}}{(1-q)(2-q)} \\ & + \frac{\left[(1-\alpha)\ddot{x}_{n-1} + \alpha \ddot{x}_n\right] x_{n-1} \Delta t^{2-q}}{(1-q)(2-q)} \\ & + \frac{3\left[(1-\alpha)\ddot{x}_{n-1} + \alpha \ddot{x}_n\right] \dot{x}_{n-1} \Delta t^{3-q}}{(1-q)(2-q)(3-q)} \\ & + \frac{3\left[(1-\alpha)\ddot{x}_{n-1} + \alpha \ddot{x}_n\right]^2 \Delta t^{4-q}}{(1-q)(2-q)(3-q)(4-q)} \end{aligned} \right\} \quad (13)$$

Thus, at t_n the nonlinear numerical scheme of the fractional-differential Eqs. (2) and (3) of nonlinear oscillators can be set up.

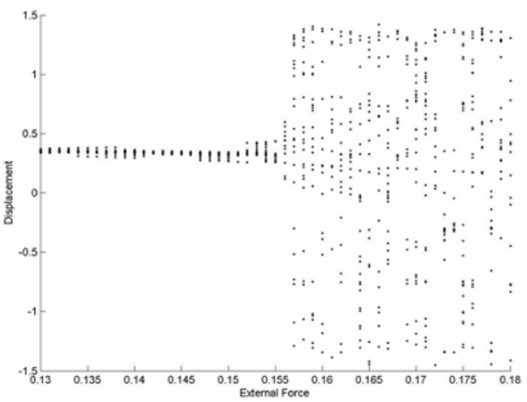
Fig. 1 illustrates the bifurcation diagrams of the Duffing-like fractional oscillator with different orders of fractional derivatives, which are solved by means of our proposed numerical scheme. It can be seen that the algorithm of the fractional derivative developed here shows the correctness and effectiveness of the nonlinear numerical scheme. A detailed discussion on the Duffing-like fractional oscillator can be found in paper [8].

3. Nonlinear dynamic behavior

The nonlinear dynamic behavior of the oscillator described in Eq. (2) is first presented here. Suppose the zero initial condition and $f(t)=FCos(\pi t)$, and set the model parameters at $c_0=25$, $c_1=0.05$, $k=(2\pi)^2$, $F=1$, $\alpha=0.5$, and $\beta=0.25$. If we change only the order of fractional differentials, we can obtain the following results as depicted in Fig. 2.



(a) q=1

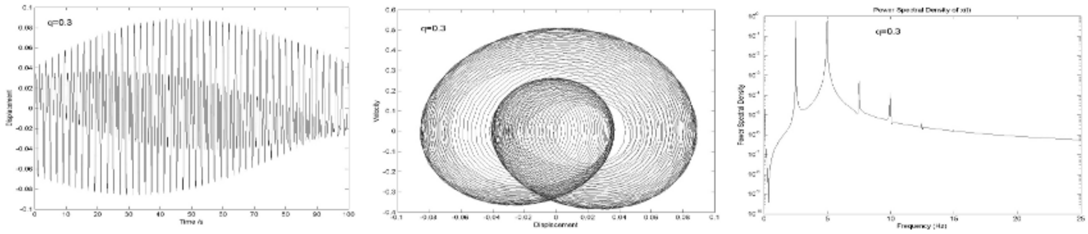


(b) q=0.75

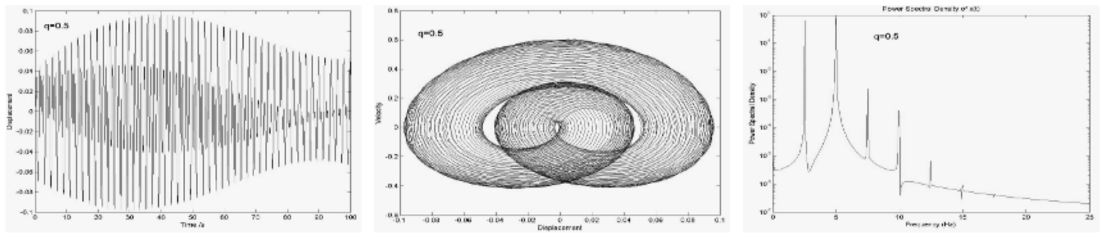
Fig. 1. Bifurcation diagram of the Duffing-like fractional oscillator.

The results show that the orders of fractional derivatives evidently affect the dynamic behavior of the nonlinear oscillator. The increase of the order of fractional differential has seen the increase of the degree of bifurcation of the oscillator. Insight into this phenomenon exposes the material damping characterization, with the values of fractional differential orders for the dissipation system modeled by fractional calculus. This conclusion is more obvious when compared with the oscillator described by the integer differential operator if we keep the other parameters unchanged, as shown in Fig. 3.

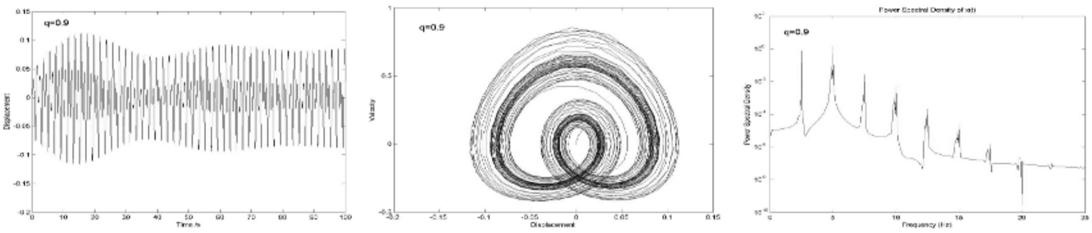
It can be seen from Fig. 3 that the dynamic behavior of the oscillator with the integer differential operator is more "chaotic," and its energy dissipation capability is weaker than the one described here by the fractional differential operator.



(a) Order of fractional differential q=0.3



(b) Order of fractional differential q=0.5



(c) Order of fractional differential q=0.9

Fig. 2. Time response, phase graph, and power spectrum of the nonlinear oscillator.

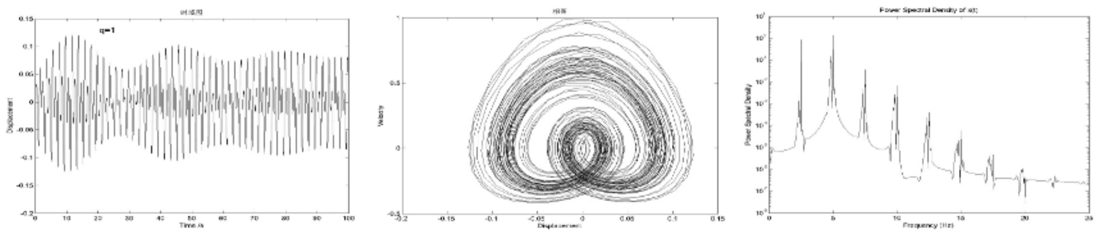


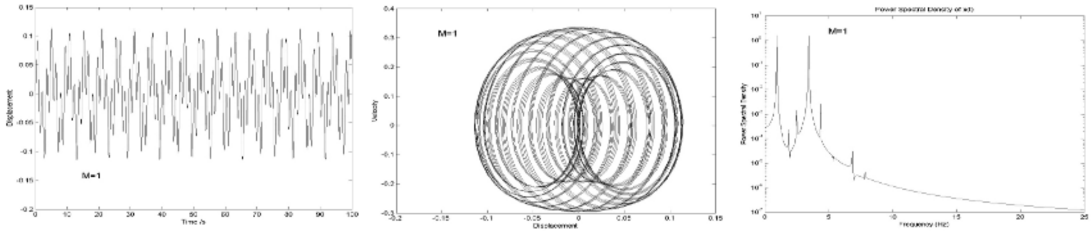
Fig. 3. Time response, phase graph, and power spectrum of the nonlinear oscillator when the order of fractional differential q=1.0.

The strength of outside excitation can certainly influence the nonlinear dynamic behavior of the oscillator. Suppose that $c_0=5$, $c_1=0$, $k=6\pi$, $q=0.5$, $\alpha=0.5$, and $\beta=0.25$, and let amplitude F change from *1 unit* to *15*. The numerical solutions can be presented as depicted in Fig. 4.

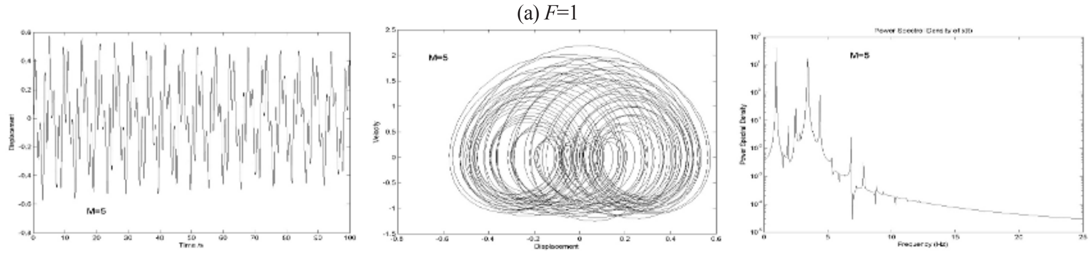
As we predicted, the larger the amplitude of excitation, the stronger the nonlinearity of the oscillator is. When the strength of the outside excitation reaches a point, the chaos of the oscillator occurs. We can also assume the twin attractors of the oscillator. Therefore, although this kind of oscillator described in Eq. (2) is constrained by the stronger damping described by the fractional calculus viscoelastic law, it can still evolve into strong and typical nonlinear dynamic behaviors, which are the same as the correspondent nonlinear oscillator with integer derivative.

If Eq. (3) of the Van der pol-like fractional differential oscillator is treated as a special case of Eq. (2), then the nonlinear dynamic behaviors will appear as illustrated in Fig. 5, as we set $\varepsilon=5$, F=10, and $\omega=1.5$.

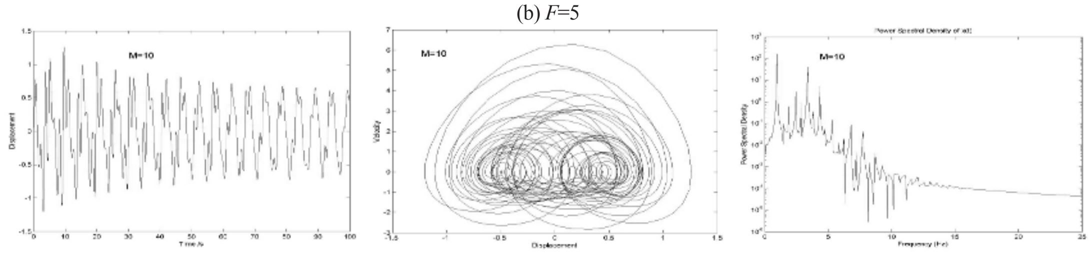
The data in Fig. 5 demonstrate that along with the change of the orders of fractional derivative from a small to a large nonlinearity of this type of oscillator is the change from strong to less complicated. In other words, the influence of the orders of fractional deriva-



(a) F=1



(b) F=5



(c) F=10

Fig. 4. Time response, phase graph, and power spectrum of the nonlinear oscillator when the order of fractional differential q=0.5.

q=0.1 q=0.3 q=0.75

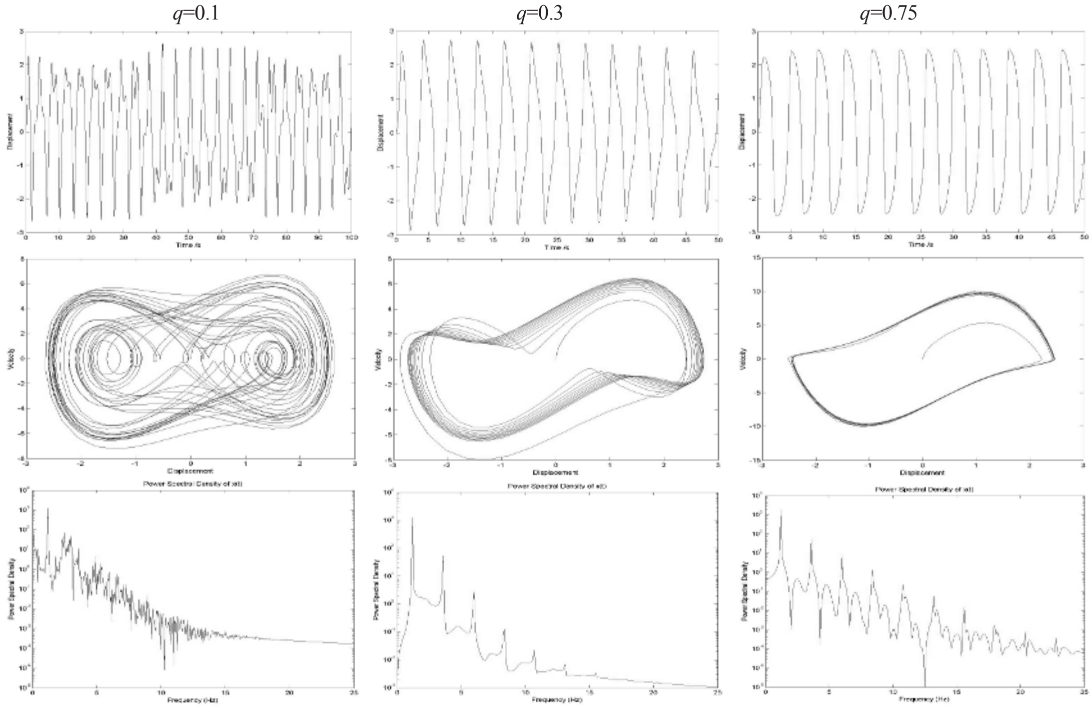


Fig. 5. Influence of the orders of fractional derivative.

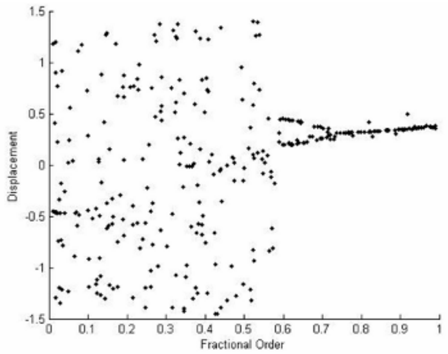


Fig. 6. Bifurcation diagram of the displacement-fractional derivative orders.

tive indeed strongly affects the dynamic behavior of the system. In fact, it can be indicated that the fraction values of the fractional derivatives reflect the characteristics of the energy dissipation of such systems as polymeric damping materials [8].

The bifurcation diagram in Fig. 6 depicts the dynamic behaviors of the fractional differential oscillator along with the changes in the order of fractional derivative. The chaotic vibration of these kinds of dynamic systems can be encountered. Again, the fraction values of the fractional derivatives influence the dynamic status of the systems they describe, and they can surely reflect the characteristics of the energy dissipation of such systems as polymeric damping materials.

4. Conclusions

The oscillator dynamic systems associated with a type of nonlinear viscoelasticity modeled by a fractional differential operator are proposed and carefully studied in this paper. The numerical scheme for the solution of the nonlinear oscillator is successfully developed. The main results of the research in this paper can be summarized as follows:

The order of fractional derivative evidently affects the dynamic behavior of the nonlinear oscillator. The fraction values of the fractional derivatives reflect the characteristics of the energy dissipation of such systems as polymeric damping materials.

The introduction of the nonlinear damping brings about not only stronger energy dissipation in some cases, but it may also incur the more complex dynamic namic behavior of the oscillator system as well in other cases.

Acknowledgement

This work is part of the project supported by the Natural Science Foundation of China (No.10872079: Shock Absorbing and Energy Dissipation Mechanism of Advanced Polymer Damping Materials).

References

[1] K. Addolfsson, Mikael Enelund, S. Larsson, Models and numerical procedures for nonlinear fractional order viscoelastics fractional differentiation and its application, Germany: Bordeaux 1 University, ISBN:3-86608-026-3 (2005), 365-376.

[2] Y. Meshaka, A. Stephane and C. Christian, Describing viscoelasticity using thermodynamic functions: link with fractional type operators, France: FDA04, 1st IFAC workshop on fractional differentiation and its application 19-21(2004).

[3] H. Nasuno and N. Shimizu, Nonlinear Statical and Dynamical Models of Fractional Derivative Viscoelastic Body, Proc. 2005 ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference. September 24-28, Long Beach, California, USA(2005).

[4] H. Z. Sun, Constitutive law of viscoelastic materials modeled by fractional-order derivatives, Master-degree thesis, JiNan University, China (2006, In Chinese).

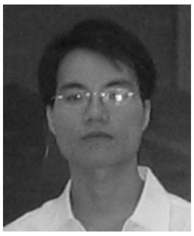
[5] Wei ZHANG and Nobuyuki SHIMIZU, Numerical Algorithm for Dynamic Problems Involving Fractional Operators, Int. J. of JSME, Ser. C, 41, NO.3, (1998)364-370.

[6] John Guckenheimer and Philip Holmes, Nonlinear oscillations, dynamical systems and bifurcations of vector fields, Fifth Printing, New York: Springer-Verlag(1997).

[7] W. Zhang, H. Xu and N. Shimizu, Numerical analysis formulation of viscoelastic solids modeled by fractional operator, ACTA MECHANICA SINICA, 36 (5) Sept. (2004) 617-621 (In Chinese).

[8] S. K. Liao and W. Zhang, Dynamics of fractional Duffing oscillator, Journal of Dynamics and Control, 6 (2) (2008) 122-125 (In Chinese).

Wei Zhang received his doctorate degree in Mechanics at Xi'an Jiaotong University in China in 1994**.** He went to the University Of Sheffield in the UK as an academic visitor from July 1986 to July 1987. He completed his post-doctorate study in Iwaki Meisei University in Japan through the support of the JSPS from May 1997 to June 1999. Dr. Zhang is currently a professor at the Department of Electronic Engineering of JiNan University in Guangzhou, China.



Shao-kai Liao received his B.E. degree in Civil Engineering at JiNan University in Guangzhou, China in 2004 and his master's degree in Engineering Mechanics at the same university in 2006. Mr. Liao was once a postgraduate student of Professor Wei Zhang, and he now works in an institute in Shenzhen City in China.

Nobuyuki Shimizu received his B.S. degree in Mechanical Engineering at Yokohama National University and his master's and doctorate degrees in Mechanical Engineering at University of Tokyo in 1968 and 1971, respectively. He went to the California Institute of Technology in California, U.S.A. as a visiting associate from October 1976 to September 1977. Dr. Shimizu is currently a professor at the Department of Mechanical and Power Engineering of Iwaki Meisei University in Japan.